Supplementary Notes for ELEN 4810 Lecture 8
Poles, Zeros and Discrete-Time LTI Systems

John Wright
Columbia University

November 17, 2025

Disclaimer: These notes are intended to be an accessible introduction to the subject, with no pretense at completeness. In general, you can find more thorough discussions in Oppenheim's book. Please let me know if you find any typos.

Reading suggestions: Oppenheim and Schafer Sections 5.1-5.7.

In this lecture, we discuss how the poles and zeros of a rational $\mathcal{Z}$-transform $H(z)$ combine to shape the frequency response $H(e^{j\omega})$.

1 Frequency Response, Phase and Group Delay

Recall that if x is the input to an LTI system with impulse response $h[n]$, we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \tag{1.1}$$

a relationship which can be studied through the $\mathcal{Z}$ transform identity

$$Y(z) = H(z)X(z) \tag{1.2}$$

(on $\mathrm{ROC}\,\{h\} \cap \mathrm{ROC}\,\{x\}$), or, if h is stable, through the DTFT identity

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}). \tag{1.3}$$

This relationship implies that magnitudes multiply and phases add:

$$\begin{aligned} |Y(e^{j\omega})| &= |H(e^{j\omega})||X(e^{j\omega})|, & (1.4)\\ \angle Y(e^{j\omega}) &= \angle H(e^{j\omega}) + \angle X(e^{j\omega}). & (1.5) \end{aligned}$$

The phase $\angle\cdot$ is only defined up to addition by an integer multiple of 2π. In the next two lectures, we will occasionally need to be more precise about the phase. The text uses the notation $\mathrm{ARG}[H(e^{j\omega})]$ for the unique value $\angle H(e^{j\omega}) + 2\pi k$ lying in $(\pi, \pi]$:

$$-\pi < \mathrm{ARG}[H] \leq \pi. \tag{1.6}$$

This gives a well-defined way of removing the 2π ambiguity in the phase. However, the graph of ARG often contains 2π discontinuities, at those points ω for which $\angle H$ passes out of $(-\pi, \pi]$.

These discontinuities are purely artifacts of choosing $(-\pi, \pi]$ as a the range of ARG. For the functions $H(e^{j\omega})$ that we consider, $H(e^{j\omega})$ varies continuously with ω (and is even a differentiable function of ω). The text therefore also defines a "continuous phase", denoted $\arg[H(e^{j\omega})]$, such that for every ω, $\arg[H(e^{j\omega})] = \angle H(e^{j\omega}) + 2\pi k$ for some $k \in \mathbb{Z}$, and $\arg[H]$ is continuous, and $\arg[H]$ "starts out" in $(-\pi, \pi]$: $-\pi < \arg[H(e^{j0})] \leq \pi$.

These tedious distinctions are necessary because we are about to do calculus on the phase. We define the *group delay* as the negative derivative of the (continuous) phase:

Definition 1.1 (Group delay)**.** *The* group delay *is defined as:*

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\arg[H(e^{j\omega})]. \tag{1.7}$$

We make three examples of increasing complexity to justify the name "group delay", and explain why this quantity is interesting.

Example I: An Ideal Delay. Suppose that $h[n] = \delta[n - \ell]$ for some $\ell \in \mathbb{Z}$. Then we know that the frequency response is $H(e^{j\omega}) = \exp(-j\omega\ell)$. Note that $|H(e^{j\omega})| = 1$ for all ω, and that

$$\arg[H(e^{j\omega})] = -\omega\ell. \tag{1.8}$$

Hence,

$$\mathrm{grd}[H(e^{j\omega})] = \ell. \tag{1.9}$$

For the ideal delay by ℓ samples, the group delay is simply the number ℓ!

Example II: Narrowband Signals. Consider a signal $s[n]$, whose DTFT $S(e^{j\omega})$ is concentrated around $\omega = 0$. If we modulate it by a complex exponential $e^{j\omega_0 n}$, we produce a signal

$$x[n] = s[n]e^{j\omega_0 n} \tag{1.10}$$

whose DTFT is concentrated around ω_0 (and $\pm\omega_0 + 2\pi\mathbb{Z}$, by 2π periodicity of the DTFT).

Let $y[n]$ be the output of a stable LTI system with impulse response $h[n]$ on input x: $y = h * x$. Using the group delay, we can make a first order approximation of the phase around ω_0:

$$\begin{aligned} \arg[H(e^{j\omega})] &\approx \arg[H(e^{j\omega_0})] - (\omega - \omega_0)\mathrm{grd}[H(e^{j\omega_0})] && (1.11) \\ &= -\phi_0 - \omega\ell, && (1.12) \end{aligned}$$

where we have defined $\phi_0 = -\arg[H(e^{j\omega_0})] - \omega_0 \mathrm{grd}[H(e^{j\omega_0})]$ and let $\ell \in \mathbb{Z}$ be the nearest integer approximation of the group delay $\mathrm{grd}[H(e^{j\omega_0})]$. So, for $\omega \approx \omega_0$, we have

$$\begin{aligned} H(e^{j\omega}) &= |H(e^{j\omega})|e^{j\arg[H(e^{j\omega})]} && (1.13) \\ &\approx |H(e^{j\omega_0})|e^{-j\phi_0}e^{-j\omega\ell}. && (1.14) \end{aligned}$$

The output of the system is

$$
\begin{aligned}
y[n] &= \mathrm{DTFT}^{-1}\left\{H(e^{j\omega})X(e^{j\omega})\right\}[n] && (1.15)\\
&\approx \mathrm{DTFT}^{-1}\left\{e^{-j\omega\ell}X(e^{j\omega})\right\}[n] && (1.16)\\
&= |H(e^{j\omega_0})|e^{-j\phi_0}\mathrm{DTFT}^{-1}\left\{e^{-j\omega\ell}X(e^{j\omega})\right\}[n] && (1.17)\\
&= |H(e^{j\omega_0})|e^{-j\phi_0}x[n-\ell]. && (1.18)
\end{aligned}
$$

Thus, *the action of an LTI system on a narrowband input $x[n]$ can be approximated by (i) a scaling by the magnitude response at ω_0, (ii) a phase shift, and (iii) a delay by a number of samples dictated by the group delay.*

Example III: Narrowband Signals. In communications applications, we often encounter signals of the form

$$x[n] = s[n]\cos(\omega_0 n) \tag{1.19}$$

where $s[n]$ is the true signal of interest and is assumed to be real, and it is modulated by a (co)-sinusoid of frequency ω_0 as part of a communication scheme. Typically, the DTFT of s is concentrated about zero – the frequencies of interest in s are much smaller than ω_0. As above, we can idealize this a bit by imagining that s is low-pass, and that the bandlimit is very small. $X(e^{j\omega})$ is concentrated about $\pm\omega_0$. To approximate the output of the system on input $x[n]$, we can develop similar approximations to $H(e^{j\omega})$ in the vicinity of ω_0 and $-\omega_0$. If we assume that $h[n]$ is real valued, then $H(e^{j\omega})$ is conjugate symmetric – the magnitude response is even and the phase response is odd. This allows us to phrase the approximation at $-\omega_0$ in terms of that at ω_0:

$$H(e^{j\omega}) \approx |H(e^{j\omega_0})|e^{j\phi_0}e^{j\omega\ell} \qquad \omega \approx -\omega_0. \tag{1.20}$$

Using these two approximations, we can derive that

$$y[n] \approx |H(e^{j\omega_0})|s[n-\ell]\cos(\omega_0 n - \phi_0 - \omega_0\ell). \tag{1.21}$$

That is to say, the action of the system on the narrowband signal x is to multiply by $|H(e^{j\omega_0})|$, introduce a phase shift of ϕ_0 in the sinusoidal component, and delay the whole thing by ℓ samples. Recall again that $\ell = \mathrm{grd}[H(e^{j\omega_0})]$ is the group delay.

So, in this example, frequencies around ω_0 are delayed by the value ℓ of the group delay at ω_0. This further justifies the name. You may wonder whether our calculations above are really meaningful – we have made quite a few approximations. It is clear that the estimate above is more precise when (i) the signal $x[n]$ is truly narrowband, so we only need to consider frequencies ω that are very close to ω_0, (ii) the magnitude of $H(e^{j\omega})|$ doesn't change too quickly with ω, and (iii) the first order expansion in the phase is accurate. It is possible to quantify how badly each of these assumptions is violated, in terms of the properties of the signal x and the system h, and work out bounds on the approximation error using calculus. We will not pursue this here, although you are welcome to think carefully about it – especially if any of the above steps seem vague or uncomfortable.

For assumption (iii) – accuracy of the first order approximation to the phase, we note that this is equivalent to the group delay being nearly constant (since the group delay is the first derivative, and the first order approximation becomes less accurate as the first derivative changes). One important class of systems is those with *linear phase*: $H(e^{j\omega}) = |H(e^{j\omega})|\exp(-j\omega\ell)$, for some constant ℓ. Such systems act on the magnitude of the Fourier transform, while introducing a delay ℓ which is constant across frequency. Often, a moderate constant delay is acceptable,[1] and achieving (nearly) linear phase is an important design goal. When we design FIR filters, we will explicitly enforce linearity of the phase; for IIR design we will seek designs which are nearly linear phase within some range of frequencies of interest.

Example IV: A Superposition of Narrowband Signals. To help our intuition on group delay, we simulated an example from the text (Section 5.1.2).

2 Some Intuition for a Single Factor

Hopefully, the previous example helped clarify the intuitive meaning of the group delay: by plotting the magnitude and group delay, we were able to predict the attenuation and delay of the various frequency components. This is obviously a useful approach, provided someone has already provided us with $H(z)$! However, if our goal is to design $H(z)$, we need to understand better how the various terms in its expression conspire to produce the result.

We will consider this question for rational $H(z)$. When the ROC contains the unit circle, this leads to a rational DTFT $H(e^{j\omega})$, which can be expressed in terms of its poles and zeros:

$$H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1-\zeta_k e^{-j\omega})}{\prod_{\ell=1}^{N}(1-\rho_\ell e^{-j\omega})}. \tag{2.1}$$

Here the ζ_k are the zeros, and the ρ_ℓ are the poles. In the following, we assume that there are no pole-zero cancelations, i.e., we do not have $\zeta_k = \rho_\ell$ for any k, ℓ.

The above expression for the DTFT is very complicated. Fortunately, if we want to understand the net effect on the phase, the group delay, or the gain (log magnitude response), we can decompose the effect into a superposition of terms that depend on a single pole or zero:

$$\begin{aligned}
20\log_{10}|H(e^{j\omega})| &= 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}|1-\zeta_k e^{-j\omega}| - \sum_{\ell=1}^{N} 20\log_{10}|1-\rho_\ell e^{-j\omega}| && (2.2)\\
\arg[H(e^{j\omega})] &= \arg\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M}\arg\left[1-\zeta_k e^{-j\omega}\right] - \sum_{\ell=1}^{n}\arg[1-\rho_\ell e^{-j\omega}] \quad + \quad 2\pi\mathbb{Z} && (2.3)\\
\operatorname{grd}\left[H(e^{j\omega})\right] &= \sum_{k=1}^{M}\operatorname{grd}\left[1-\zeta_k e^{-j\omega}\right] - \sum_{\ell=1}^{N}\operatorname{grd}\left[1-\rho_\ell e^{-j\omega}\right]. && (2.4)
\end{aligned}$$

The first term above is known as the *gain*, and is expressed in *decibels*. This logarithmic quantity is preferred both for perceptual reasons, and because the individual factors enter in an additive manner. Using the gain, phase and group delay, we can study the net effect of the rational DTFT $H(e^{j\omega})$ (2.1). Because the phase and the log-magnitude are additive, we can make substantial progress by just understanding a single factor $1-\zeta_k e^{-j\omega}$.

[1] For example, the human auditory system is not too sensitive to absolute delays. Interestingly, we are very sensitive to relative delays, or relative phase, between the signals obtained by our two ears – this is how we guess which direction a sound is coming from. So, for stereo systems, we need very tight control on the relative delay.

Write $\zeta_k = r\exp(j\theta)$, with $r, \theta \in \mathbb{R}$, so $1 - \zeta_k e^{-j\omega} = 1 - re^{j\theta}e^{-j\omega}$. A bit of manipulation and calculus (verify these!) show that

$$\begin{aligned}
\textbf{Gain:} \quad & 20\log_{10}\left|1 - re^{j\theta}e^{-j\omega}\right| &=& \; 10\log_{10}\left[1 + r^2 - 2r\cos(\omega - \theta)\right] & (2.5)\\
\textbf{Phase:} \quad & \arg\left[1 - re^{j\theta}e^{-j\omega}\right] &=& \; \arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right] & (2.6)\\
\textbf{Group delay:} \quad & \operatorname{grd}\left[1 - re^{j\theta}e^{-j\omega}\right] &=& \; \frac{r^2 - r\cos(\omega-\theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}. & (2.7)
\end{aligned}$$

We plot these functions for various choices of r and θ.

3 Real Impulse Responses

In the previous section, we saw how to think about rational transfer functions through terms defined by the individual poles and zeros, and how to interpret the individual terms by diagrams in the complex plane. We next discuss several important categories of systems, in which the poles and zeros interact in interesting ways.

In the following, it will be useful to have in mind the specific symmetries in the pole-zero diagram for systems with a real impulse response:

Lemma 3.1. *Suppose that $h[n]$ is real. Then for every $z \in \mathrm{ROC}\{h\}$, $H(z) = H^*(z^*)$.*

Proof. We calculate:

$$\begin{aligned}
H(z) &= \sum_n h[n]z^{-n} & (3.1)\\
&= \sum_n h^*[n]z^{-n} & (3.2)\\
&= \sum_n h^*[n]\left(z^{*-n}\right)^* & (3.3)\\
&= \left(\sum_n h[n]z^{*-n}\right)^* & (3.4)\\
&= H^*(z^*), & (3.5)
\end{aligned}$$

as claimed. □

This implies that for real $h[n]$, the zeros occur in conjugate pairs, as do the poles:

Proposition 3.2. *For rational h, (i) whenever z is a zero, so is z^*, and (ii) whenever z is a pole, so is z^*.*

4 Systems with a Given Magnitude Response

When designing frequency-selective filters (low-pass, band-pass, ect.), often our first thought is how to shape the magnitude response $|H(e^{j\omega})|$ of the filter. For a given choice of $|H(e^{j\omega})|$, there are an arbitrarily large number of possible choices of $\angle H(e^{j\omega})$. However, if we restrict our attention to systems with rational $H(z)$, there are *some* constraints between the magnitude and the phase.

Lemma 4.1. *For a stable system,* $|H(e^{j\omega})|^2 = [H(z)H^*(1/z^*)]\big|_{z=e^{j\omega}}$.

Proof. Notice that

$$\begin{aligned} H^*(z) &= \left(\sum_n h[n]z^{-n}\right)^* && (4.1) \\ &= \sum_n h^*[n](z^*)^{-n}, && (4.2) \end{aligned}$$

and so

$$H^*(1/z^*) = \sum_n h^*[n]z^n, \qquad (4.3)$$

giving $H^*(1/z^*)\big|_{z=e^{j\omega}} = \sum_n h[n]e^{j\omega n} = [H(e^{j\omega})]^*$. Since $|H(e^{j\omega})|^2 = H(e^{j\omega})[H(e^{j\omega})]^*$, the claim follows. □

Set $C(z) = H(z)H^*(1/z^*)$. Notice that if $H(z)$ is rational:

$$H(z) = \alpha \frac{\prod_{i=1}^{M}(1-\zeta_i z^{-1})}{\prod_{j=1}^{N}(1-\rho_j z^{-1})}, \qquad (4.4)$$

then

$$C(z) = |\alpha|^2 \frac{\prod_{i=1}^{M}(1-\zeta_i z^{-1})(1-\zeta_i^* z)}{\prod_{j=1}^{N}(1-\rho_j z^{-1})(1-\rho_j^* z)}. \qquad (4.5)$$

The zeros of $C(z)$ are the ζ_i and $1/\zeta_i^*$; the poles are ρ_j and $1/\rho_j^*$.

Suppose that $\rho_j = re^{j\phi}$, with $r \in \mathbb{R}_+$ is the polar form of ρ_j. Then $1/\rho_j^* = \frac{1}{re^{-j\phi}} = \frac{1}{r}e^{j\phi}$. That is to say, *the pole* $1/\rho_j$ *has the same angle as* ρ_j*, but is "reflected" about the unit circle, from magnitude* r *to magnitude* $1/r$.

Thus, the poles of $C(z)$ each come in "reflected pairs" $(\rho_j, 1/\rho_j^*)$; the zeros also come in reflected pairs. One element of each pair arises from $H(z)$; the other comes from $H^*(1/z^*)$. In general, there is no way to decide which is which. However, if we know ahead of time that the system should be causal and stable, then its ROC must extend outward from the largest magnitude pole, and must include the unit circle. In this case, we know that whichever of ρ_j and $1/\rho_j^*$ lies inside the unit circle, must be the one that comes from $H(z)$. This is enough to tell us the poles of $H(z)$, but does not tell us which of the zeros belong to $H(z)$ and $H^*(1/z^*)$. *Different assignments of the zeros of* $C(z)$ *to* $H(z)$ *and* $H^*(1/z^*)$ *yield different causal, stable systems* $H(z)$ *with the same magnitude response* $|H(e^{j\omega})|$. Beyond this ambiguity in the assignment of the zeros, there is another inevitable ambiguity: for any system $H_{ap}(z)$ which satisfies $|H_{ap}(e^{j\omega})| = 1$ for all ω, $|H(e^{j\omega})| = |H(e^{j\omega})H_{ap}(e^{j\omega})|$. Hence, $H(z)$ and $H(z)H_{ap}(z)$ cannot be distinguished based on the magnitude response alone. A system H_{ap} with this property is called an *all-pass* system.

5 All-Pass Systems

Definition 5.1 (All-pass system)**.** *An* all-pass *system is an LTI system whose frequency response $H(e^{j\omega})$ exists, and for which $|H(e^{j\omega})| = 1$ for all ω.*[2]

All-pass systems do not change the magnitude of $X(e^{j\omega})$ – they only affect the phase. In the last lecture, we saw an example of how drastically changing the phase in a nonlinear manner can change the qualitative properties of a signal – changing the phase can delay various frequency components. Rational all-pass systems can built out of individual components of the form

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}, \tag{5.1}$$

where $a \in \mathbb{C}$. To see that this corresponds to an all-pass system, note that for every ω,

$$\begin{aligned}
|H(e^{j\omega})| &= \left| \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \right| && (5.2)\\
&= \left| e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} \right| && (5.3)\\
&= \left| \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} \right| && (5.4)\\
&= \left| \frac{(1 - ae^{-j\omega})^*}{1 - ae^{-j\omega}} \right| && (5.5)\\
&= 1. && (5.6)
\end{aligned}$$

Rational all-pass transfer functions can be built as products of terms of form (5.1). If $h[n]$ is real, then any a that is not purely real must occur together with its conjugate a^*:

$$(\textbf{General rational all-pass } H(z)) \qquad H(z) = e^{j\phi} \prod_{k=1}^{M} \frac{z^{-1} - a_k^*}{1 - a_k z^{-1}}, \qquad a_k \in \mathbb{C},$$

$$(\textbf{Rational all-pass } H(z)\textbf{, for real } h[n]) \qquad H(z) = \pm \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{\ell=1}^{M_c} \frac{(z^{-1} - e_\ell^*)(z^{-1} - e_\ell)}{(1 - e_\ell z^{-1})(1 - e_\ell^* z^{-1})}, \qquad d_k \in \mathbb{R}$$

where M_r is the number of real pole-zero pairs, and $2M_c$ is the number of non-real pole-zero pairs.

It is clear that an all-pass system cannot have poles or zeros on the unit circle. Since the ROC for a causal impulse response $h[n]$ with $H(z)$ rational extends outward from the largest magnitude pole, for causal all-pass systems we can say a bit more:

Proposition 5.2. *If $H(z) = e^{j\phi} \prod_k \frac{z^{-1} - a_k^*}{1 - a_k z^{-1}}$ is the transfer function for a* causal *all-pass system, then all of the poles a_k lie strictly inside the unit circle: $|a_k| < 1$.*

Moreover,

[2] The text extends the definition to allow systems with $H(e^{j\omega})| = \alpha$ for all ω – the magnitude response is constant, but not required to be exactly equal to one.

Proposition 5.3. *Let* $H(z) = e^{j\phi} \prod_k \frac{z^{-1}-a_k^*}{1-a_k z^{-1}}$ *be the transfer function for a causal, all-pass system. Then* $\mathrm{grd}[H(e^{j\omega})] > 0$ *for all* ω.

Proof. Consider a single factor of the form $H_k(z) = \frac{z^{-1}-a_k^*}{1-a_k z^{-1}}$. Using the polar form $a_k = re^{j\theta}$ ($r, \theta \in \mathbb{R}$), and calculus, we obtain

$$\mathrm{grd}\left[H_k(e^{j\omega})\right] \;=\; \frac{1-r^2}{|1-re^{j\theta}e^{-j\omega}|^2} > 0, \tag{5.7}$$

where we have used the fact that $r < 1$, which follows from the previous proposition. Since $\mathrm{grd}[H] = \sum_k \mathrm{grd}[H_k]$, $\mathrm{grd}[H(e^{j\omega})] > 0$. □

Since the group delay is the negative derivative of the phase, this implies that for a causal all-pass system, the phase is a decreasing function of ω. When $h[n]$ is real, we can say a bit more:

Proposition 5.4. *If* $H(z) = \prod_k \frac{z^{-1}-d_k}{1-d_k z^{-1}} \prod_\ell \frac{(z^{-1}-e_\ell^*)(z^{-1}-e_\ell)}{(1-e_\ell z^{-1})(1-e_\ell^* z^{-1})}$ *is the transfer function of an all-pass system with real impulse response* $h[n]$, *then* $\arg[H(e^{j\omega})] < 0$ *for all* $\omega > 0$.

Proof. Plugging $z = 1$ into the general form of a rational all-pass $H(z)$ with real $h[n]$,

$$H(z) = \prod_{k=1}^{M_r} \frac{z^{-1}-d_k}{1-d_k z^{-1}} \prod_{\ell=1}^{M_c} \frac{(z^{-1}-e_\ell^*)(z^{-1}-e_\ell)}{(1-e_\ell z^{-1})(1-e_\ell^* z^{-1})}, \tag{5.8}$$

we obtain $H(e^{j0}) = H(1) = 1$. So $\arg[H(e^{j0})] = 0$. Because the group delay is positive, and group delay is the negative derivative of phase, the derivative of phase with respect to ω is negative. Since $\arg[H(e^{j0})] = 0$ and $(d/d\omega)\arg[H(e^{j\omega})] < 0$, $\arg[H(e^{j\omega})] < 0$ for every $\omega > 0$. □

6 Minimum Phase Systems

In this section we discuss a family of systems known as *minimum phase systems*. There are several equivalent ways of defining a minimum phase system. We give a definition and interpret its properties, and then circle back to explain the name "minimum phase".

Minimum phase systems have well-behaved inverse systems. Recall that if an LTI system with impulse response $h[n]$ is stable, then for any bounded input $x[n]$, the output $y[n]$ is well defined (and bounded). An inverse system, with impulse response $h_i[n]$, satisfies the property that for every bounded input x, $h_i * (h * x)$ is well-defined and equal to x:

$$h_i * (h * x)[n] = x[n]. \tag{6.1}$$

In $\mathcal{Z}$-domain, $H_i(z)H(z) = 1$ on $\mathrm{ROC}[h_i] \cap \mathrm{ROC}[h]$. It is possible to write down LTI systems that do not have an inverse – consider for example, a low-pass filter. Minimum phase systems are systems which have a well-structured inverse:

Definition 6.1 (Minimum phase)**.** *An LTI system is* minimum phase *if it stable, causal, and has a stable, causal inverse system.*

If $H(z)$ is rational, then $H_i(z) = H(z)^{-1}$. So, the poles of $H(z)$ become the zeros of $H_i(z)$, and the zeros of $H(z)$ become the poles of $H_i(z)$. For $H(z)$ to be stable and causal, the poles must lie inside the unit circle; for $H_i(z)$ to be stable and causal, *its* poles must lie inside the unit circle, and so the zeros of $H(z)$ must also lie inside the unit circle:

Proposition 6.2. *An LTI system with rational transfer function is minimum phase if and only if all of its poles and zeros lie strictly inside the unit circle.*

Suppose that we have a transfer function $H(z) = H_1(z)(z^{-1} - \beta^*)$, where all of the zeros of $H_1(z)$ lie inside the unit circle, but the zero β^{-1} lies outside the unit circle. Then we can write

$$H(z) = \underbrace{H_1(z)(1 - z^{-1}\beta)}_{\text{Minimum Phase}} \times \underbrace{\frac{z^{-1} - \beta^*}{1 - z^{-1}\beta}}_{\text{All Pass}}. \tag{6.2}$$

The second factor $\frac{z^{-1}-\beta^*}{1-z^{-1}\beta}$ is *all pass*. For the first factor $H_1(z)(1 - z^{-1}\beta)$, we have "reflected" the zero at $1/\beta^*$ to lie at β, inside the unit circle.

Theorem 6.3 (Min-phase, all-pass decomposition)**.** *Every LTI system with rational transfer function $H(z)$ with no poles or zeros on the unit circle can be expressed as $H(z) = H_{min}(z)H_{ap}(z)$, where H_{min} is minimum phase, and H_{ap} is all-pass.*

Proof. Repeat the process described above for each pole and zero lying outside the unit circle. □

Minimum phase systems satisfy a number of "minimality" properties, which state that out of all systems with a given magnitude response, the minimum phase system minimizes the group delay and related quantities.

Proposition 6.4 (Minimum group delay)**.** *If $H_{\min}(z)$ is rational, minimum phase, and $H(z)$ is a rational transfer function for a causal, stable system with the same magnitude response ($|H(e^{j\omega})| = |H_{\min}(e^{j\omega})|$ for all ω), then*

$$\operatorname{grd}\left[H_{\min}(e^{j\omega})\right] \le \operatorname{grd}\left[H(e^{j\omega})\right] \qquad \forall\,\omega. \tag{6.3}$$

Proof. Using the minimum-phase all-pass decomposition, we can write $H(z) = H_{\min}(z)H_{ap}(z)$, where $H_{\min}$ is minimum-phase, and H_{ap} is all-pass. We have

$$\begin{aligned} \operatorname{grd}\left[H(e^{j\omega})\right] &= \operatorname{grd}\left[H_{\min}(e^{j\omega})\right] + \underbrace{\operatorname{grd}\left[H_{ap}(e^{j\omega})\right]}_{\ge 0,\ \forall\,\omega} && (6.4) \\ &\ge \operatorname{grd}\left[H_{\min}(e^{j\omega})\right]. && (6.5) \end{aligned}$$

□

The idea of the proof of the minimum group delay property is very simple: the group delay of a product is simply the sum of the group delays of the individual terms, and the group delay of an all-pass filter is nonnegative. So, multiplying by an all-pass transfer function $H_{ap}(z)$ never decreases the group delay. You can consult Section 5.6.3 of the text for discussion of a related property of the *phase lag* $-\arg[H(e^{j\omega})]$ of a minimum phase system. The phase lag property is actually the motivation for the term "minimum phase".

Similarly, minimum phase systems minimize the "energy delay":

Proposition 6.5. *Suppose that $h_{\min}$ is the impulse response for a minimum phase, causal, stable system with rational transfer function, and h is the impulse response for a causal, stable system with rational transfer function and the same magnitude response $|H_{\min}(e^{j\omega})| = |H(e^{j\omega})|$. Then for each $n \geq 0$,*

$$\sum_{m=0}^{n} |h[m]|^2 \;\leq\; \sum_{m=0}^{n} |h_{\min}[m]|^2. \tag{6.6}$$

7 Linear Phase

In many applications, it is desirable to design filters that affect the magnitude of the Fourier transform of the input in some way – for example, by attenuating certain frequency components, while affecting the phase as little as possible. Unfortunately, for causal systems, it is not possible to achieve zero phase. Instead, as a design goal, we can try to make the phase as close to linear as possible. We saw in previous lectures that the negative derivative of the phase, i.e., the group delay, indicates how frequency components of the input are delayed.

Definition 7.1 (Linear Phase)**.** *A system with frequency response $H(e^{j\omega})$ has linear phase if its frequency response can be expressed as*

$$H(e^{j\omega}) = |H(e^{j\omega})| \times e^{-j\omega\alpha}, \qquad -\pi < \omega \leq \pi. \tag{7.1}$$

That is to say, the phase $\omega\alpha$ is a *linear* function of frequency ω. The simplest example of a linear phase system is an ideal delay $h[n] = \delta[n - n_d]$. For the ideal delay, we have $H(e^{j\omega}) = e^{-j\omega n_d}$, and so the phase is linear, with slope given by the delay amount $\alpha = n_d$. We can consider more general frequency responses of the form

$$H(e^{j\omega}) = e^{-j\omega\alpha} \tag{7.2}$$

where α is not necessarily an integer. This is the Fourier transform of a sequence $h[n]$ with

$$h[n] = \frac{\sin(\pi(n-\alpha))}{\pi(n-\alpha)}, \qquad n \in \mathbb{Z}, \tag{7.3}$$

and

$$y[n] = x * h[n] = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin(\pi(n-k-\alpha))}{\pi(n-k-\alpha)}. \tag{7.4}$$

In our discussion of sampling, we gave a very clean interpretation to this system: x is converted to continuous time using an ideal discrete-to-continuous converter with period T. The resulting continuous time signal is then delayed by time αT, and then resampled with period T to produce $y[n]$.

The effect of a linear phase system with frequency response $H(e^{j\omega})$ given by (7.1) can be thought of in two stages: we first amplify or attenuate the magnitude of the Fourier transform according to $|H(e^{j\omega})|$, and then delay the resulting signal by α. Linear phase systems (and their cousins, generalized linear phase systems, which we discuss next) often satisfy interesting symmetries. For example, if $2\alpha \in \mathbb{Z}$, and the impulse response $h[n]$ is real, then $h[n]$ is symmetric about α:

Proposition 7.2. *If a linear phase system $H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}$ has real impulse response $h[n]$, and $2\alpha \in \mathbb{Z}$, then*

$$h[2\alpha - n] = h[n], \qquad \forall\, n. \tag{7.5}$$

Proof. Write $\varphi(\omega) = |H(e^{j\omega})|$. If $h[n]$ is real, $\varphi(\omega) = \varphi(-\omega)$. Using the inverse DTFT, we have

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\omega)e^{-j\omega\alpha}e^{j\omega n}d\omega && (7.6)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\omega)e^{-j\omega(\alpha-n)}d\omega. && (7.7)\end{aligned}$$

Moreover, with $\nu = -\omega$,

$$\begin{aligned} h[2\alpha-n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\omega)e^{-j\omega\alpha}e^{j\omega(2\alpha-n)}d\omega && (7.8)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\omega)e^{j\omega(\alpha-n)}d\omega && (7.9)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\nu)e^{-j\nu(\alpha-n)}d\nu && (7.10)\\ &= h[n]. && (7.11)\end{aligned}$$

□

If 2α is not an integer, $h[n]$ may not be symmetric. So, it is not true that every linear phase system has a symmetric impulse response. However, many linear phase systems encountered are symmetric, or nearly symmetric.[3]

8 Generalized Linear Phase

Consider the causal, $M+1$ point *moving average*, with

$$h[n] = \begin{cases} \frac{1}{M+1} & 0 \le n \le M \\ 0 & \text{else.} \end{cases} \tag{8.1}$$

Its frequency response is

$$H(e^{j\omega}) = \underbrace{\frac{1}{M+1}\frac{\sin\left[\omega(M+1)/2\right]}{\sin(\omega/2)}}_{A(e^{j\omega})} e^{-j\omega M/2}. \tag{8.2}$$

This is a product of a real valued function $A(e^{j\omega})$ and a linear-phase complex exponential. At first glance, it looks like a linear phase system. However, it is not linear phase. When $A(e^{j\omega})$ is negative, the phase is shifted by π:

$$H(e^{j\omega}) = |A(e^{j\omega})|e^{-j\omega M/2+\pi(1-\mathrm{sign}(A(e^{j\omega})))/2}. \tag{8.3}$$

[3]Here, the restriction $-\pi < \omega \le \pi$ in (7.1) is important. It is not difficult to show that if we demand $H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}$ for all $\omega \in \mathbb{R}$, this forces $2\alpha \in \mathbb{Z}$ – use 2π periodicity of the DTFT.

Definition 8.1 (Generalized Linear Phase)**.** *A system has* generalized linear phase *if its frequency response has the form*

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega+j\beta}, \qquad -\pi < \omega \leq \pi. \tag{8.4}$$

where $A(e^{j\omega})$ *is real.*

If $A(e^{j\omega})$ is continuous, then the group delay is constant, except at points ω for which $A(e^{j\omega}) = 0$:

$$\text{grd}\left[H(e^{j\omega})\right] = \alpha \qquad \forall\, \omega \text{ s.t. } A(e^{j\omega}) \neq 0. \tag{8.5}$$

When $A(e^{j\omega}) = 0$, the phase can be discontinuous, and the group delay may not be defined. For example, in the moving average, $A(e^{j\omega})$ changes sign as ω moves through any point ω at which $A(e^{j\omega}) = 0$, causing the phase to change by π. Away from these points, the group delay is simply $M/2$. This makes generalized linear phase an appealing design target. We will describe four simple families of FIR generalized linear phase systems, which arise in filter design.[4]

Symmetries of Generalized Linear Phase Systems. The text derives a necessary condition for $h[n]$ to be the impulse response of a generalized linear phase system with parameters α and β – namely,

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\omega(n-\alpha)+\beta] = 0 \qquad \forall\, \omega. \tag{8.6}$$

This condition imposes a family of "symmetries" on $h[n]$. It can be proved as follows: for our generalized linear phase system, write

$$H(e^{j\omega}) = A(e^{j\omega})e^{j(\beta-\alpha\omega)} = A(e^{j\omega})\cos(\beta-\alpha\omega) + A(e^{j\omega})j\sin(\beta-\alpha\omega) \tag{8.7}$$

and

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\cos(\omega n) - j\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n). \tag{8.8}$$

Equating real and imaginary parts of these two expressions for H, we obtain

$$\frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)} = \frac{-\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)}{\sum_{n=-\infty}^{\infty} h[n]\cos(\omega n)}. \tag{8.9}$$

Cross multiplying and using the angle addition identity $\sin(\theta+\phi) = \sin(\theta)\cos(\phi)+\sin(\phi)\cos(\theta)$, we obtain (8.6). Fixing various choices of α and β, we can obtain more concrete necessary conditions on the impulse response. For example, if $\beta = 0$ and $M = 2\alpha \in \mathbb{Z}$, the necessary condition

$$\sum_{n=-\infty}^{\infty} h[n]\sin(\omega(n-\alpha)) = 0 \tag{8.10}$$

can be shown to imply that h is symmetric about α. Different choices of α and β in (8.6) impose different families of conditions on $h[n]$. In the next section, we will look at four families of FIR generalized linear phase systems, classified based on their symmetries.

[4] IIR generalized linear phase systems are much tricker to design, although they do exist.

9 Four Canonical Causal FIR Generalized Linear Phase Systems

We describe four canonical types of causal FIR generalized linear phase systems.

Type I: Symmetric Impulse Response, Odd Length. A Type I FIR generalized linear phase system has a real impulse response $h[n]$, which is symmetric, in the sense that $h[n] = h[M-n]$, $0 \le n \le M$, where M an *even integer* (so $M/2$ is an integer). For a Type I system,

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} && (9.1)\\
&= \sum_{n=0}^{\frac{M}{2}-1} h[n]\left(e^{-j\omega n} + e^{-j\omega(M-n)}\right) + h[M/2]e^{-j\omega M/2} && (9.2)\\
&= e^{-j\omega M/2}\left\{\sum_{n=0}^{\frac{M}{2}-1} h[n]\left(e^{-j\omega(n-M/2)} + e^{-j\omega(M/2-n)}\right) + h[M/2]\right\} && (9.3)\\
&= e^{-j\omega M/2}\left\{\sum_{n=0}^{\frac{M}{2}-1} h[n]\times 2\cos(\omega(n-M/2)) + h[M/2]\right\}. && (9.4)
\end{aligned}
$$

We can rewrite this slightly by setting $a[0] = h[M/2]$ and $a[k] = 2h[(M/2)-k]$ $(k = 1, \ldots, M/2)$, to obtain

$$
H(e^{j\omega}) = e^{-j\omega M/2}\sum_{k=0}^{M/2} a[k]\cos(\omega k). \tag{9.5}
$$

This demonstrates $H(e^{j\omega})$ as a product of a linear-phase exponential term and a real function $A(e^{j\omega})$. Hence, Type I FIR systems have generalized linear phase.

Type II: Symmetric Impulse Response, Even Length. A Type II FIR generalized linear phase system has a real impulse response $h[n]$, which is symmetric, in the sense that $h[n] = h[M-n]$, $0 \le n \le M$, where M is an *odd integer*. For a Type II system,

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} && (9.6)\\
&= e^{-j\omega M/2}\sum_{n=0}^{\frac{M-1}{2}} h[n]\left(e^{-j\omega(n-M/2)} + e^{-j\omega(M/2-n)}\right) && (9.7)\\
&= e^{-j\omega M/2}\sum_{n=0}^{\frac{M-1}{2}} 2h[n]\cos(\omega(n-M/2)). && (9.8)
\end{aligned}
$$

With a bit of manipulation, we obtain

$$H(e^{j\omega}) = e^{-j\omega M/2} \sum_{k=1}^{(M+1)/2} b[k] \cos(\omega(k - 1/2)). \tag{9.9}$$

where $b[k] = 2h[(M+1)/2 - k]$. This again demonstrates $H(e^{j\omega})$ as a linear-phase exponential term and a real function. Hence, Type II FIR systems also have generalized linear phase.

Type III: Antisymmetric Impulse Response, Odd Length. A Type III FIR generalized linear phase system has a real impulse response $h[n]$, which is antisymmetric, in the sense that $h[n] = -h[M-n]$, $0 \leq n \leq M$, where M is an *even integer*. As above, we calculate

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n] e^{-j\omega n} && (9.10)\\
&= \sum_{n=0}^{M/2-1} h[n] \left(e^{-j\omega n} - e^{-j\omega(M-n)} \right) + h[M/2] e^{-j\omega M/2} && (9.11)\\
&= e^{-j\omega M/2} \left\{ \sum_{n=0}^{M/2-1} h[n] \left(e^{-j\omega(n-M/2)} - e^{-j\omega(M/2-n)} \right) + h[M/2] \right\} && (9.12)\\
&= e^{-j\omega M/2} \left\{ \sum_{n=0}^{M/2-1} h[n] \times 2j \sin(-\omega(n - M/2)) + h[M/2] \right\}. && (9.13)
\end{aligned}$$

With some manipulation, this becomes

$$H(e^{j\omega}) = j e^{-j\omega M/2} \sum_{k=1}^{M/2} c[k] \sin(\omega k) \tag{9.14}$$

with $c[k] = 2h[(M/2) - k]$. This again demonstrates $H(e^{j\omega})$ as a product of a constant phase ($j = e^{j\pi/2}$), a linear-phase exponential ($e^{-j\omega M/2}$) and a real-valued function of ω. So, Type III systems are again generalized linear phase.

Type IV: Antisymmetric Impulse Response, Even Length. A Type IV FIR generalized linear phase system has a real impulse response $h[n]$, which is antisymmetric, i.e., $h[n] = -h[M-n]$, with M an *odd integer*. Then

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} \tag{9.15}$$

$$= \sum_{n=0}^{\frac{M-1}{2}} h[n]\left(e^{-j\omega n} - e^{-j\omega(M-n)}\right) \tag{9.16}$$

$$= e^{-j\omega M/2} \sum_{n=0}^{\frac{M-1}{2}} h[n]\left(e^{-j\omega(n-M/2)} - e^{-j\omega(M/2-n)}\right) \tag{9.17}$$

$$= e^{-j\omega M/2} \sum_{n=0}^{\frac{M-1}{2}} 2jh[n]\sin\left(-\omega(n-M/2)\right). \tag{9.18}$$

Again, putting this in a slightly nicer form, we get

$$H(e^{j\omega}) = je^{-j\omega M/2} \sum_{k=1}^{(M+1)/2} d[k]\sin(\omega(k-1/2)), \tag{9.19}$$

demonstrating $H(e^{j\omega})$ as a product of a phase $j = e^{j\pi/2}$, a linear phase exponential $e^{-j\omega M/2}$, and a real-valued function of ω.

10 Zeros of Type I-IV FIR Filters

Each type of generalized linear phase filter described above has a characteristic pattern of zeros. For Type I and II systems, which have symmetric impulse responses, note that

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} \tag{10.1}$$

$$= \sum_{n=0}^{M} h[M-n]z^{-n} \tag{10.2}$$

$$= z^{-M}\sum_{n=0}^{M} h[M-n]z^{M-n} \tag{10.3}$$

$$= z^{-M}H(z^{-1}). \tag{10.4}$$

Hence, if z_0 is a zero of $H(z)$, $0 = z_0^{-M}H(z_0^{-1})$, and so z_0^{-1} is also a zero of $H(z)$. Because $h[n]$ is real, z_0^* and $(z_0^{-1})^*$ are also zeros. So, if $z_0 = re^{j\theta}$, there are zeros at

$$z_0 = re^{j\theta}, \qquad z_0^{-1} = \frac{1}{r}e^{-j\theta}, \qquad z_0^* = re^{-j\theta}, \qquad 1/z_0^* = \frac{1}{r}e^{j\theta}. \tag{10.5}$$

If z is real (i.e., $\theta = 0$ or $\theta = \pi$), and $|r| \neq 1$, there will be two distinct zeros above, at r and $1/r$.

Type	Symmetries	M	Characteristic zeros
I	Symmetric $h[n] = h[M-n]$	M even	none
II	Symmetric $h[n] = h[M-n]$	M odd	-1
III	Antisymmetric $h[n] = -h[M-n]$	M even	$1, -1$
IV	Antisymmetric $h[n] = -h[M-n]$	M odd	1

Table 1: **Properties of Generalized Linear Phase FIR Filters**

For Type III and IV systems, which have anti-symmetric impulse responses, very similar considerations apply. Note that

$$\begin{aligned} H(z) &= \sum_{n=0}^{M} h[n] z^{-n} && (10.6) \\ &= -\sum_{n=0}^{M} h[M-n] z^{-n} && (10.7) \\ &= -z^{-M} \sum_{n=0}^{M} h[M-n] z^{M-n} && (10.8) \\ &= -z^{-M} H(z^{-1}). && (10.9) \end{aligned}$$

Hence, again, if z_0 is a zero of $H(z)$, so is z_0^{-1}. Again, because $h[n]$ is real-valued, this forces zeros at z_0^* and $1/z_0^*$ as well.

Zeros at 1 and -1. Type III and IV systems have the following characteristic property: evaluating (10.9) at $z = 1$, we obtain $H(1) = -H(1)$. This implies that $H(1) = 0$. So $z_0 = 1$ *is always a zero of a Type III or Type IV FIR system*. Similarly, for a Type III system, M is an even integer, and so $(-1)^{-M} = 1$. Evaluating (10.9) at $z = -1$, we obtain that $H(-1) = -H(-1)$, and so $z_0 = -1$ is a zero. Similarly, for a Type II system, M is odd, $(-1)^{-M} = -1$; evaluating (10.4) at $z_0 = -1$, we obtain $H(-1) = -H(-1)$, implying that $z_0 = -1$ is a zero. So $z_0 = -1$ *is always a zero of a Type II or Type III FIR system*.

These characteristic zero locations have implications of the appropriateness of these structures for various filtering tasks. For example, a Type II system would be a poor choice for a high-pass filter, since its magnitude response is always zero at $\omega = \pi$. A Type IV system could not function as a low-pass filter, since its magnitude response is zero at $\omega = 0$. A Type III system has zeros at both -1 and 1, and so it cannot serve as a low-pass filter or a high-pass filter, but might be appropriate for constructing bandpass systems.